Independence (Linear and Affine)

Linear Algebra

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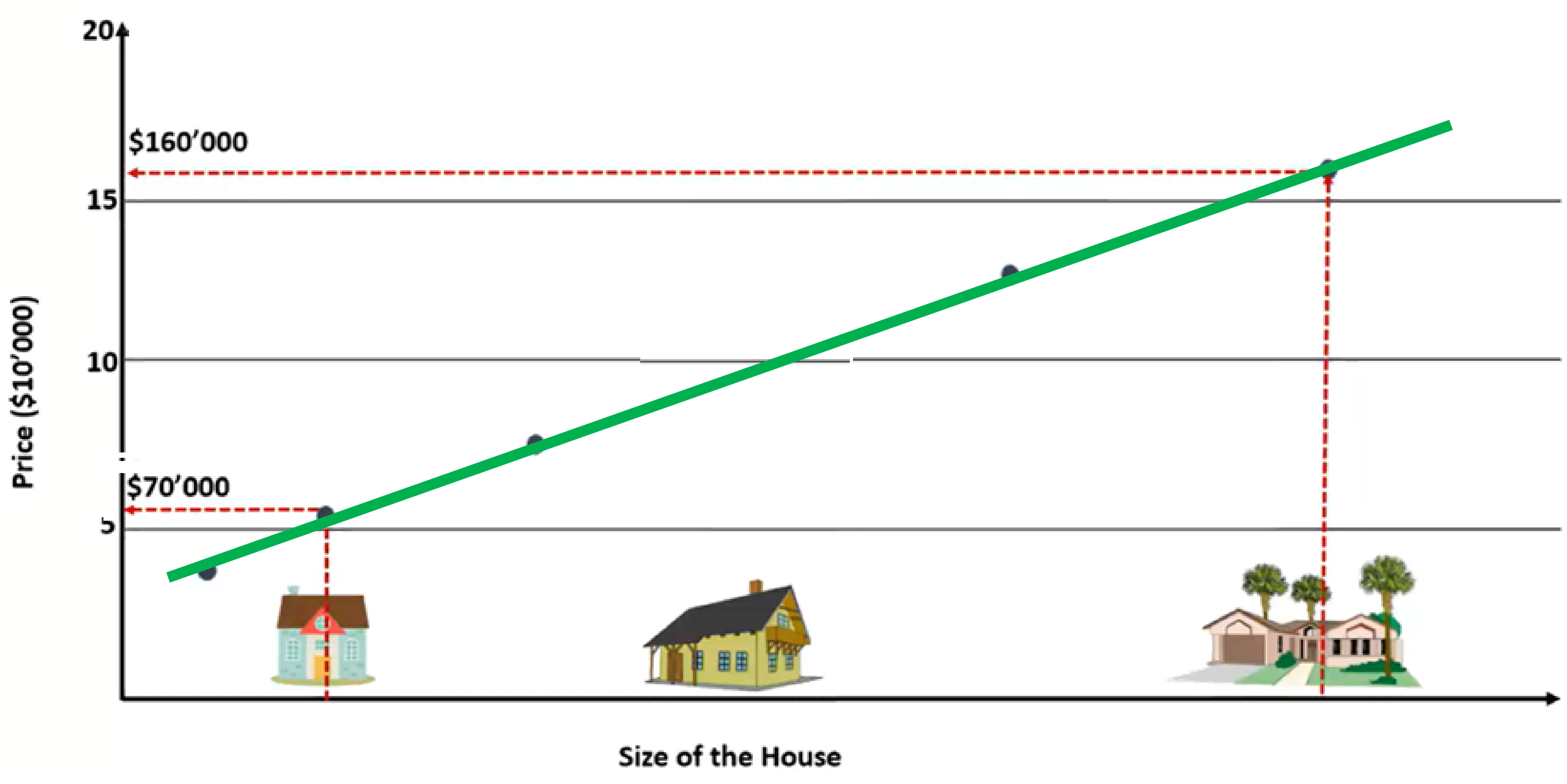
Overview

- Introduction
- Linear Independence
- Functions Linearly Independent
- Polynomials Linearly Independent
- Affine Combination
- Affine Independence

Introduction

Introduction

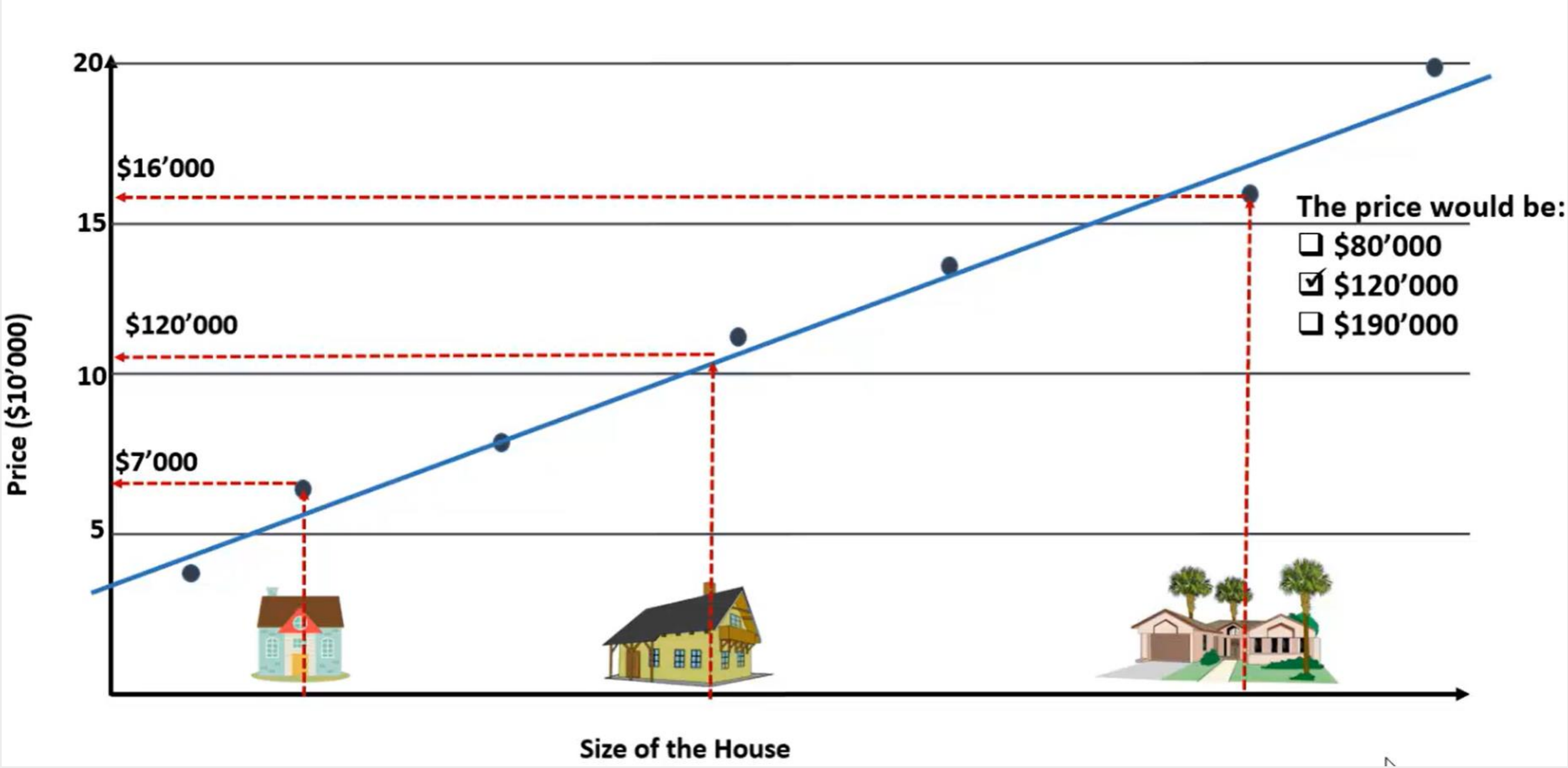
Linear Equation



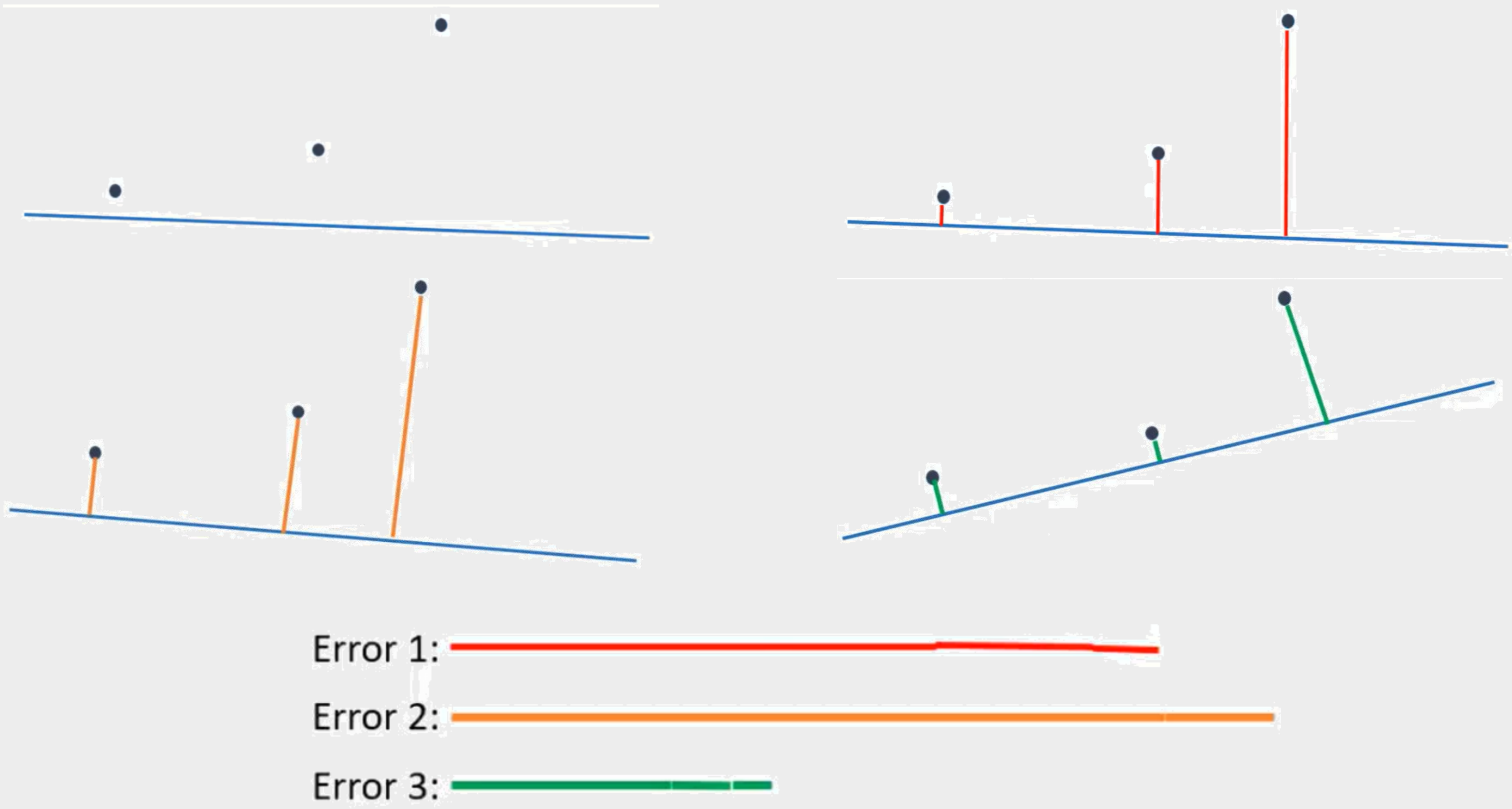
Linear Independent

- #Room
- Size
- #Bedroom
- Age
- Address features: Street, Alley, ⋯
- Size of part1, part2, part3, part4
- Floors
- #Bathrooms
- ⋯

Least Squares Error Correction

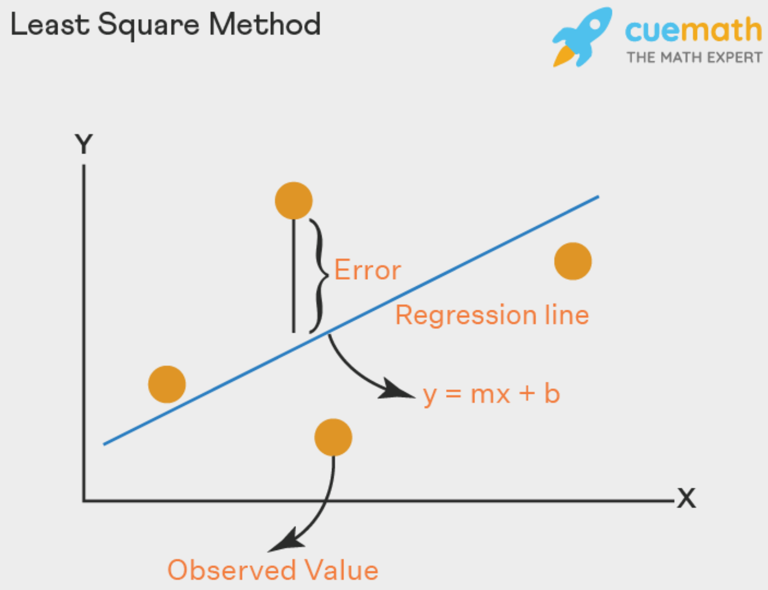


Least Squares Error Correction



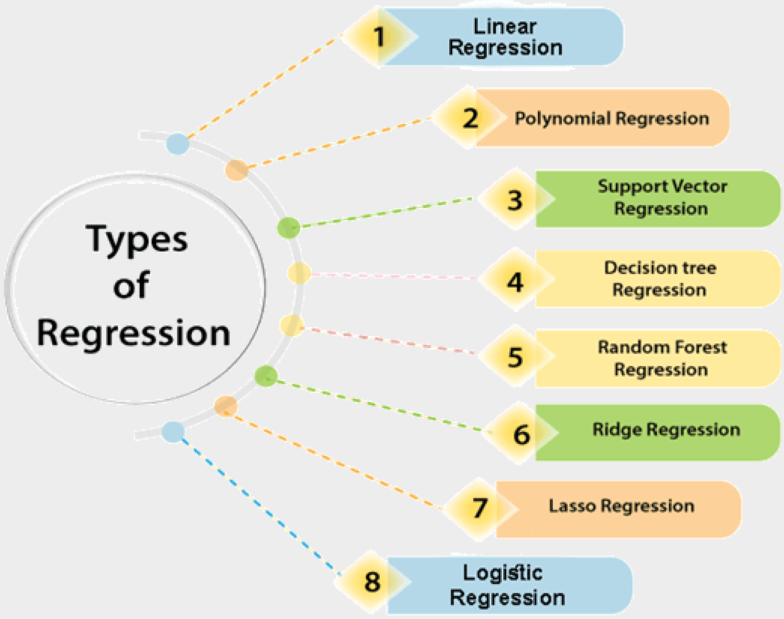
- Objective: $\hat{y} = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n + b$

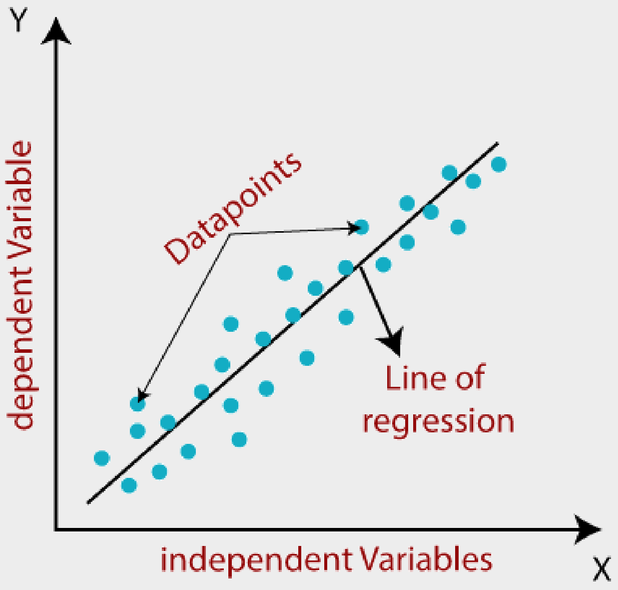
$$\min ||y - \hat{y}||$$



- $Ax = b \quad \rightarrow \quad x = A^{-1}b$ Inverse of Matrix/Pseudo Inverse

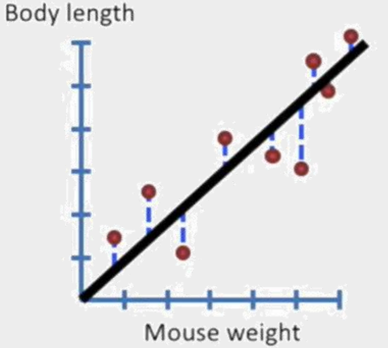
- Regression




Y
dependent Variable
Datapoints
Line of
regression
independent Variables
X

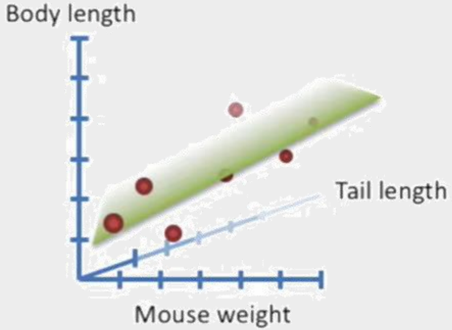
Linear Algebra and Machine Learning Application

Simple regression



y = y-intercept + slope x

Multiple regression



y = y-intercept + slope x + slope z

Linear Independence

Linear Independence (Algebra)

Definition

Dependent

- For at least one $\lambda \neq 0$ $\quad 0 = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n, \quad \lambda \in \mathbb{R}$
- A set of vectors is dependent if at least one vector in the set can be expressed as a linear weighted combination of the other vectors in that set.

Definition

Independent

- Only when all $\lambda_i = 0$ $\quad 0 = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n, \quad \lambda \in \mathbb{R}$
- No vector in the set is a linear combination of the others (has only the trivial solution)

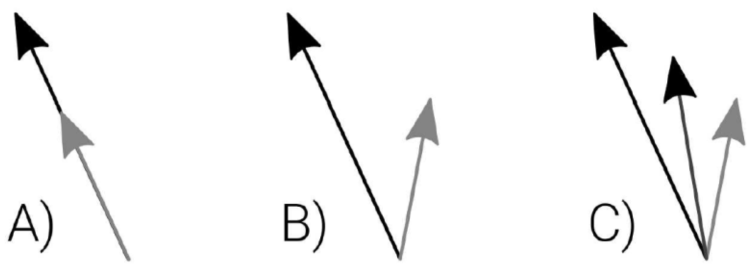
Linear Independence (Geometry)

Definition

A set of vectors is linear independent if the subspace dimensionality (its span) equals the number of vectors.

Example

- vectors spans?



Geometric sets of vectors in $\mathbb{R}^2$

Example

Example

- Let $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, and\ v_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$

- a) $v_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

  b) $v_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

Properties

Theorem

Any set of vectors that contains the zeros vector is guaranteed to be linearly dependent.

Proof

Characterization of Linearly Dependent sets

Theorem

An indexed set $S = \{v_1, \dots, v_n\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others.

In fact, if S is linearly dependent and $v_1 \neq 0$, then **some** v_j (with $j > 1$) is a linear combination of the preceding vectors, $v_1, \dots, v_{j-1}$.

Proof

- Does not say that every vector
- Does not say that every vector in a linearly dependent set is a linear combination of the preceding vectors. A vector in a linearly dependent set may fail to be a linear combination of the other vectors.

Proof

If some v_j in S equals a linear combination of the other vectors, then v_j can be subtracted from both sides of the equation, Producing a linear dependence relation with a nonzero weight (-1) on v_j. [For instance, if $v_1 = c_2v_2 + c_3v_3$, then $0 = (-1)v_1 + c_2v_2 + c_3v_3 + 0v_4 + \cdots + 0v_n$.] Thus S is linearly dependent.

Conversely, suppose S is linearly dependent. If v_1 is zero, then it is a (trivial) linear combination of the other vectors in S. Otherwise, $v_1 \neq 0$, and there exist weights $c_1, \dots, c_n$ not all zero, such that

$$c_1v_1 + c_2v_2 + \cdots + c_pv_n = 0$$

Proof

Let j be the largest subscript for which $c_j \neq 0$. If $j = 1$, then $c_1 v_1 = 0$, which is impossible because $v_1 \neq 0$. So $j > 1$ and

$$c_1 v_1 + \cdots + c_j v_j + 0 v_{j+1} + 0 v_n = 0$$

$$c_j v_j = -c_1 v_1 - \cdots - c_{j-1} v_{j-1}$$

$$v_j = \left(-\frac{c_1}{c_j}\right) v_1 + \cdots + \left(-\frac{c_{j-1}}{c_j}\right) v_{j-1}$$

Properties

- The vectors coming from the vector form of the solution of a matrix equation Ax = 0 are linearly independent

Example

- Vectors related to x_2 and x_3 are linear independent.
- Columns of A related to to x_2 and x_3 are linear dependent.
- Span$\{A_1, A_2, A_3\} = Span\{A_1\}$

$$\mathrm{A} = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Important

- If a collection of vectors is linearly dependent, then any superset of it is linearly dependent.
- Any nonempty subset of a linearly independent collection of vectors is linearly independent.

Properties

Theorem

- ❑ Any set of $p > n$ vectors in $\mathbb{R}^n$ is necessarily dependent.
- ❑ Any set of $p \le n$ vectors in $\mathbb{R}^n$ could be linearly independent.

Proof

$$
\begin{array}{cc}
 & p \\
n & \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}
\end{array}
$$

Example

Example

$a.\quad \begin{bmatrix} 1 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix}$

$b.\quad \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 8 \end{bmatrix}$

$c.\quad \begin{bmatrix} -2 \\ 4 \\ 6 \\ 10 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ -9 \\ 15 \end{bmatrix}$

Linear Dependent Properties

- Suppose vectors $v_1, \dots, v_n$ are linearly dependent:

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$$

with $c_1 \neq 0$. Then:

$$span\{v_1, \dots, v_n\} = span\{v_2, \dots, v_n\}$$

- When we write a vector space as the space of a list of vectors, we would like that list to be as short as possible. This can achieved by iterating.

Theorem

Suppose x is linear combination of linearly independent vectors

$v_1, \ldots, v_n$:

$$x = \beta_1 v_1 + \cdots + \beta_n v_n$$

The coefficients $\beta_1, \ldots, \beta_n$ are unique.

Proof

Conclusion

Important

- Step 1: Count the number of vectors (call that number p) in the set and compare to n in $\mathbb{R}^n$. As mentioned earlier, if $\mathrm{p} > n$, then the set is necessarily dependent. If $\mathrm{p} \leq n$ then you have to move on to step 2.
- Step 2: Check for a vector of all zeros. Any set that contains the zeros vector is a dependent set.
- The rank of a matrix is the estimate of the number of linearly independent rows or columns in a matrix.

Functions Linearly Independent

Functions Linearly Independent

- Let f(t) and g(t) be differentiable functions. Then they are called **linearly dependent** if there are nonzero constants c_1 and c_2 with

$$c_1 f(t) + c_2 g(t) = 0$$

for all t. Otherwise they are called **linearly independent**.

Example

Linearly dependent or independent?

- Functions $f(t) = 2\sin^2 t$ and $g(t) = 1 - cos^2 t$
- Functions $\{\sin^2 x, \cos^2 x, \cos(2x)\} \subset \mathcal{F}$

Polynomials Linearly Independent

Example

Are $(1-x),(1+x),x^2$ linearly independent?

Affine Combination

Combinations

- For vectors $x_1, x_2, \ldots, x_k$: any point y is a **linear combination** of them iff:

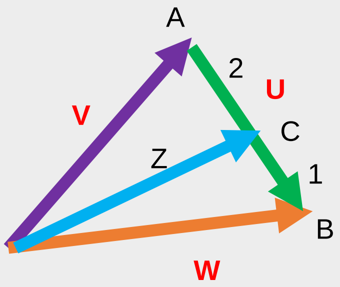
$$y = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_\mathrm{i} x_i \qquad \forall i, \alpha_\mathrm{i} \in \mathbb{R}$$

- Instead of being positive, if we put the restriction that α_i's sum up to 1, it is called an **affine combination**

$$y = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_\mathrm{i} x_i \qquad \forall i,\ \alpha_\mathrm{i} \in \mathbb{R}, \sum_i \alpha_\mathrm{i} = 1$$

Affine Combinations (Geometry)

- Linear combination and Affine combination (no origin, independent of the concepts of distance and measure of angles, keeping only the properties related to parallelism and ratio of lengths for parallel line segments)
- Affine combination of two vectors
- Affine combination of z



Affine Combination

Theorem

A point y in $\mathbb{R}^n$ is an affine combination of $v_1, \dots, v_p$ in $\mathbb{R}^n$ if and only if $y - v_i$ is a linear combination of the translated points $v_1 - v_i, v_2 - v_i, \dots, v_p - v_i$

Proof?

Example

Find a vector equation and parametric equations of the plane in $\mathbb{R}^4$ that passes through $(-17, 6, 29, 0)$, $(-13, 3, 25, -2)$ and $(-15, 6, 25, -1)$.

Affine Independence

Affine Independence

Definition

An indexed set of points $\{v_1, \dots, v_k\}$ in $\mathbb{R}^n$ is **affinely dependent** if there exists real numbers $c_1, \dots, c_k$, not all zero, such that

$$c_1 + \cdots + c_k = 0 \quad \textit{and} \quad c_1 v_1 + \cdots + c_k v_k = 0$$

Otherwise, the set is **affinely independent.**

Conclusion

- How to find affine dependent from linear dependent definition and affine combination
- Uniqueness of affine combination of affinely independent set.
- Linear dependence relation with affine dependence

Affine Independence

Note

Given an indexed set $S = \{v_1, \ldots, v_p\}$ in $\mathbb{R}^n$, with $p \geq 2$, the following statements are logically equivalent. That is, either they are all true statements or they are all false.

a. S is affinely dependent.

b. One of the points in S is an affine combination of other points in S.

c. The set $\{v_2 - v_1, \ldots, v_p - v_1\}$ in $\mathbb{R}^n$ is linearly dependent.

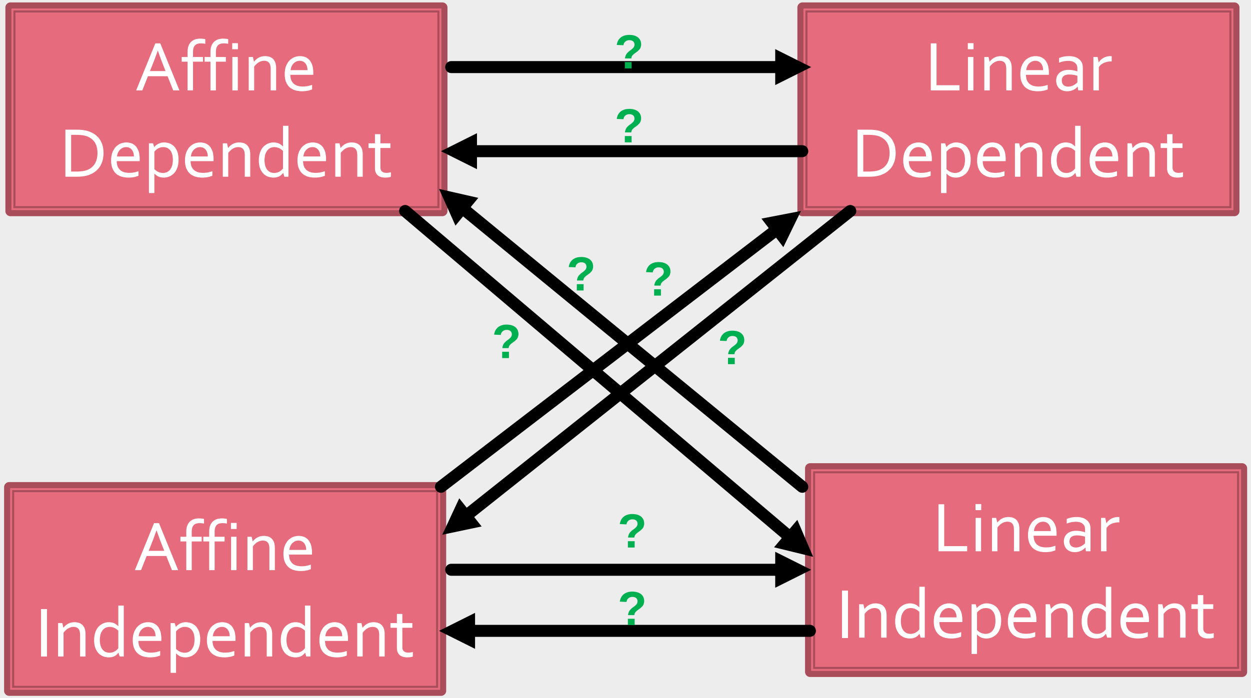
$\mathbb{R}^n$ contains at most $n + 1$ affinely independent points

Example

Example

Let $v_1 = \begin{bmatrix}1\\3\\7\end{bmatrix}, v_2 = \begin{bmatrix}2\\7\\6.5\end{bmatrix}, v_3 = \begin{bmatrix}0\\4\\7\end{bmatrix}$, and $v_4 = \begin{bmatrix}0\\14\\6\end{bmatrix}$, and let $S = \{v_1, \dots, v_4\}$. Is S affinely dependent?

Conclusion : Linear and Affine



Linearly Independent Sets versus Spanning Sets

Span	Linearly Independent
Want many vectors in small space	Want few vectors in big space
Adding vectors to list only helps	Deleting vectors from list only helps
Suppose that $v_1, \ldots, v_k$ are columns of A, now we have: AX= b has solution $\Leftrightarrow b \in span\{v_1, \ldots, v_k\}$	Suppose that $v_1, \ldots, v_k$ are columns of A, now we have: AX = 0 has only trivial solution(X=0) $\Leftrightarrow v_1, \ldots, v_k$ are linearly independent.

Reference

- Page 97 LINEAR ALGEBRA: Theory, Intuition, Code
- Page 213: David Cherney,
- Page 54: Linear Algebra and Optimization for Machine Learning